

## MULTI INSTANTON CALCULUS ON ALE SPACES

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**Abstract**

We study SYM gauge theories living on ALE spaces. Using localization formulae we compute the prepotential (and its gravitational corrections) for  $SU(N)$  supersymmetric  $\mathcal{N} = 2, 2^*$  gauge theories on ALE spaces of the  $A_n$  type. Furthermore we derive the Poincaré polynomial describing the homologies of the corresponding moduli spaces of self-dual gauge connections. From these results we extract the  $\mathcal{N} = 4$  partition function which is a modular form in agreement with the expectations of  $SL(2, \mathbb{Z})$  duality.

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## 1 Introduction

In the past two years localization techniques have proved to be very useful for the computation of non perturbative effects in gauge theories [1]-[8]. Besides being very powerful, these techniques are also very flexible and can be applied in a variety of different contexts. This work, in which we compute non perturbative effect (i.e. the prepotential) for ALE manifolds of the  $A_p$  type is an explicit demonstration of this. The viewpoint that allows for such unifying treatment is that of D-branes. In fact the bound state of  $k$  D(-1) and  $n$  D3 branes describes, in the  $\alpha' \rightarrow 0$  limit, the moduli space,  $\mathcal{M}$ , of gauge connections for the supersymmetric Yang-Mills gauge theory (SYM) with sixteen supercharges [9, 10, 11]. This prototype example is the starting point for further generalizations. The presence of the

D3 branes naturally suggests to consider the original ten dimensional space  $\mathbb{R}^{10}$  as  $\mathbb{R}^4 \otimes \mathbb{R}^6$  and the original symmetry group  $SO(10)$  as  $SO(4) \times SO(6) \sim SU(2)_L \times SU(2)_R \times SU(4)$ . These spaces can then be modded by some discrete  $\mathbb{Z}_p$  group [12]. Modding the first  $\mathbb{R}^4$  by embedding the  $\mathbb{Z}_p$  into one of the two  $U(1)$ 's of  $SU(2)_L \times SU(2)_R$ , we get ALE spaces, while if we embed  $\mathbb{Z}_p$  into one of the two  $SU(2)$ 's in the maximal subalgebra of  $SU(4)$  we find certain quiver gauge theories. The latter case will be the subject of a separate publication [13].

For some of the present authors, the interest in such models was sparked by their relation with the AdS/CFT correspondence. It is in fact well known that in the  $\mathcal{N} = 4$  SYM the space-time geometry  $AdS_5 \times S^5$  is replicated in the moduli space of gauge connections [11]. The same holds true for the cases with lower supersymmetries in which large  $N$  geometries of the type  $AdS_5 \times S^5/\mathbb{Z}_2$ ,  $AdS_5 \times S^3$ ,  $AdS_5 \times S^1$  and  $AdS_5 \times S^5/\mathbb{Z}_p$  [14, 15, 16] were recovered. At last these examples were reconsidered in [17] from the D brane viewpoint: according to the type of probe used to test the space-time geometry, the  $AdS_5 \times S^1$  (pure  $\mathcal{N} = 2$  SYM),  $AdS_5 \times S^5/\mathbb{Z}_p$  (quiver case) and  $AdS_5/\mathbb{Z}_p \times S^5$  (ALE case) cases were recovered at finite  $n$ . The present paper draws largely from this last reference and from [18] in which SYM theories on ALE instantons were studied as exact solutions of the four dimensional heterotic string equations of motion with constant dilaton and zero torsion. The techniques of the two last mentioned reference can be very profitably incorporated into the scheme of the localization described in [2, 3].

For localization to happen the gauge theories of our interest need to be deformed in a suitable. For these four dimensional theories such deformation is provided by a rigid rotation under the torus  $T_\epsilon = U(1) \times U(1)$  acting as  $(z_1, z_2) \rightarrow z_\epsilon = (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$  where  $z_1, z_2 \in \mathbb{C}^2$  are complex coordinates on the Euclidean space time. Moreover the moduli space of gauge connections must be made compact by introducing a regularization,  $\zeta$ . The blow down ALE is defined via an orbifold projection with  $\Gamma$  inside  $T_\epsilon$  [17, 18]. This implies in particular that fixed points under  $T_\epsilon$  are automatically invariant under  $\Gamma$  and therefore the analysis on  $\mathbb{R}^4$  adapts easily to the ALE case. In addition localization provides us with a powerful tool to compute the homologies of  $\mathcal{M}$  as we will explain later following

[19].

All of the results in the present paper are obtained for blown down ALE spaces and in the limit of vanishing  $\zeta$ . Still our results are valid for the full ALE due to the topological nature of the quantities we compute [19, 1, 2].

This is the plan of the paper: in section 2 we recall some details of the solution of  $\mathcal{N} = 2, 2^*$  SYM theories on  $\mathbb{R}^4$  that will be important for the following. In this very section we also briefly recall the construction of gauge connections on ALE manifold which appeared in [20]. In section 3 we study the cohomologies of the moduli spaces of self dual gauge connections on ALE manifolds and compute the  $\mathcal{N} = 4$  partition function which turns out to be a modular form. This is a strong check for the conjectured invariance (in the “weak” sense described in [21]) of  $\mathcal{N} = 4$  under electro-magnetic duality. Finally in section 4 we compute, for arbitrary values of the winding number, the prepotential for a  $\mathcal{N} = 2, 2^*$  SYM theory living on an ALE manifold. To make these results more transparent we write down the lowest terms in the expansion of the prepotential giving also the gravitational corrections.

## 2 Preliminaries

### 2.1 Localization on the ADHM manifold

The ADHM construction can be seen as a way to construct non flat hyperkähler manifolds of a given dimension starting from completely flat manifolds of dimension greater than the given one. The coordinates of the flat manifolds are organized in the ADHM matrix  $\Delta = a + bx$ . Due to the symmetries of the ADHM construction (we will later come back to this point at greater length) we may choose the matrix  $b$  so that it does not contain any moduli. Then for the gauge group  $SU(n)$ , the matrix  $\Delta$  can be written as

$$\Delta = a + bz = \begin{pmatrix} J & I^\dagger \\ B_1 & -B_2^\dagger \\ B_2 & B_1^\dagger \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \quad (2.1)$$

Here  $z_{1,2}$  (which in (2.1) are meant to be multiplied by a unit  $[k] \times [k]$  matrix) parameterize the position in the base space  $\mathbb{R}^4$  while  $m = \{J, I^\dagger, B_1, B_2\}$  are coordinates on the flat

$4k^2 + 4kn$ -dimensional hyperkähler manifold  $M = \mathbb{R}^{4k^2 + 4kn}$ . More precisely  $B_{1,2}$  and  $J, I^\dagger$  are  $[2k] \times [2k]$  and  $[n] \times [2k]$  dimensional matrices respectively. They can be thought of as homomorphisms  $B_{1,2} : V \times Q \rightarrow V$  and  $J : V \rightarrow W \times \Lambda^2 Q$ ,  $I : W \rightarrow V$ .  $V, W$  are  $[k]$  and  $[n]$  dimensional spaces respectively.  $Q$  is the 2 dimensional chiral spin space.  $1 \oplus \Lambda^2 Q \equiv 1 \oplus Q \wedge Q$  is the antichiral spin space. A self-dual field strength is built out of the ADHM connection

$$A_\mu = \bar{U}(x) \partial_\mu U(x). \quad (2.2)$$

with  $U$  a  $[2k+n] \times [n]$  matrix satisfying  $\bar{\Delta}U = 0$ . Given the three complex structures  $J_{ab}^i$  where  $i = 1, 2, 3$  and  $a, b = 1, \dots, \dim M$ , we can build the 2-forms  $\omega^i = J_{ab}^i dm^a \wedge dm^b$ . The real forms  $\omega^i$  allow one to define a  $(2, 0)$  and a  $(1, 1)$  form

$$\begin{aligned} \omega_{\mathbb{C}} &= \text{Tr } dB_1 \wedge dB_2 + \text{Tr } dI \wedge dJ, \\ \omega_{\mathbb{R}} &= \text{Tr } dB_1 \wedge B_1^\dagger + \text{Tr } dB_2 \wedge dB_2^\dagger + \text{Tr } dI \wedge dI^\dagger - \text{Tr } dJ^\dagger \wedge dJ. \end{aligned} \quad (2.3)$$

The ADHM data is invariant under

$$\begin{aligned} B_1 &\rightarrow T_{\epsilon_1} T_\phi B_1 T_\phi^{-1}, \\ B_2 &\rightarrow T_{\epsilon_2} T_\phi B_2 T_\phi^{-1}, \\ I &\rightarrow T_\phi I T_a^{-1}, \\ J &\rightarrow T_\epsilon T_a J T_\phi^{-1} \end{aligned} \quad (2.4)$$

with  $T_\phi = e^{i\phi} \in U(k)$ ,  $T_a = \text{diag}(e^{ia_1}, \dots, e^{ia_n}) \in \text{SU}(n)$ ,  $T_{\epsilon_{1,2}} = e^{i\epsilon_{1,2}} \in U(1)^2$ . The transformations  $U(n) \times U(1)^2$  describe the gauge and Lorentz symmetries of the theory while  $U(k)$  parameterizes the redundancy of the ADHM construction. Having fixed a basis  $\{e_a\}$  of  $\mathfrak{g} = U(k)$ , the condition that the Lie derivative annihilates the two forms  $\omega^i$ ,  $\mathcal{L}_{e_a} \omega^i = 0$ , leads to conserved quantities. Using a complex notation for the momenta  $f_\xi^i = f_a^i e^a$ , we get

$$\begin{aligned} f_{\mathbb{C}} &= [B_1, B_2] + IJ \\ f_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J. \end{aligned} \quad (2.5)$$

The hypersurface  $f_{\mathbb{C}} = f_{\mathbb{R}} = 0$ , modded out by the  $U(k)$  symmetries, is the ADHM moduli space  $\mathcal{M}$ .

In the non commutative case in which we allow  $[z_1, \bar{z}_1] = -\zeta/2, [z_2, \bar{z}_2] = -\zeta/2$  [22], the above condition becomes  $f_{\mathbb{C}} = 0, f_{\mathbb{R}} = \zeta$ . For  $\zeta \neq 0$  the resulting space is compact. The localization formula described below is valid only for compact spaces and therefore  $\zeta \neq 0$  will be always understood.

In the D-brane picture where instantons are thought of as D(-1)-branes superposed over D3-branes, the homomorphisms are realized by open strings connecting the  $k$  D(-1)-branes and  $n$  D3-branes and the ADHM constraints arise as D and F flatness condition (with  $\zeta$  the Fayet-Iliopoulos term) on the effective  $U(k)$  gauge theory living in the D-instanton.

Multi instanton corrections to correlators in four-dimensional gauge theories can be cast in the form of an integral over the ADHM manifold just described. Integrals over the ADHM manifold are in general hard to handle, but for particular deformations of the ADHM manifold they greatly simplify due to localizations. The physical quantities can be extracted from the final result after turning off the deformations in an suitable way. Here we briefly describe the localization formula.

Localization in the ADHM moduli space is based on the vector field  $\xi^*$ , the *fundamental vector field* associated with the group element  $\xi \in U(1)^{n-1} \times U(1)^2$ , that generates the one-parameter group  $e^{t\xi}$  of transformations on  $\mathcal{M}$ . The vector field is parameterized by the elements  $T_a, T_{\epsilon_{1,2}}$  in the Cartan of the ADHM symmetry group representing vevs and gravitational deformations. In the presence of such deformation it is possible to see that the action of  $\mathcal{N} = 2$  SYM is invariant under a deformed BRST charge operator  $Q^*$ ,  $\xi^* = 1/2\{Q^*, Q^*\}$ , and it can interpreted as a closed equivariant form [3]. This BRST charge implements the action of supersymmetry and of the symmetries (2.4) on  $\mathcal{M}$  and it can be identified with the equivariant derivative  $d_\xi = d + i_{\xi^*}$  with  $i_{\xi^*}$  the contraction using the vector field  $\xi^*$ . The localization formula is now

$$\int_{\mathcal{M}} \alpha(\xi) = (-2\pi)^{n/2} \sum_{x_0} \frac{\alpha_0(\xi)(x_0)}{\det^{\frac{1}{2}} \mathcal{L}_{x_0}} \quad (2.6)$$

where  $\alpha(\xi)$  is an equivariant form ( $e^{-S}$  in our case where  $S$  is the SYM action in  $\mathcal{M}$ ),  $\alpha_0(\xi)$  its zero degree part and  $\mathcal{L}_{x_0} : T_{x_0}\mathcal{M} \rightarrow T_{x_0}\mathcal{M}$  the map generated by the vector

field  $\xi^*$  evaluated at the critical points  $x_0$ . See [3] for a more detailed explanation of the notations employed here. The critical points are defined to be the fixed points  $\xi^*(x_0) = 0$  of the vector field  $\xi^*$  up to a diagonalizable  $U(k)$  gauge transformation given by  $T_\phi = \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_k})$ ,

From the infinitesimal version of (2.4) we find

$$\begin{aligned} (\phi_{IJ} + \epsilon_\ell) B_{IJ}^\ell &= 0 \\ (\phi_I - a_\alpha) I_{I\alpha} &= 0 \\ (-\phi_I + a_\alpha + \epsilon) J_{\alpha I} &= 0 \end{aligned} \tag{2.7}$$

with  $\epsilon = \epsilon_1 + \epsilon_2$  and  $\phi_{IJ} = \phi_I - \phi_J$ .

The solutions of (2.7) can be put in one to one correspondence with a set of  $n$  Young tableaux  $(Y_1, \dots, Y_n)$  with  $k = \sum_\alpha k_\alpha$  boxes distributed between the  $Y_\alpha$ 's. The boxes in a  $Y_\alpha$  diagram are labelled either by the instanton index  $I_\alpha = 1, \dots, k_\alpha$  or by a pair of integers  $j_\alpha, i_\alpha$  denoting the vertical and horizontal position respectively in the Young diagram. The explicit solutions to (2.7) can then be written as

$$\phi_{I_\alpha} = \phi_{i_\alpha j_\alpha} = a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2 \tag{2.8}$$

and  $J = B_\ell = I = 0$  except for the components  $B_{1(i_\alpha j_\alpha), (i_\alpha - 1 j_\alpha)}$ ,  $B_{2(i_\alpha j_\alpha), (i_\alpha j_\alpha - 1)}$ ,  $I_{1,1}$ ,  $I_{k_\alpha + 1, \alpha + 1}$   $\alpha = 1, \dots, n$ . At the critical points the spaces  $V, W$  become  $T_\epsilon$  and  $T_a$  modules allowing the decomposition

$$\begin{aligned} V &= \sum_{(i_\alpha, j_\alpha) \in Y_\alpha} T_{a_\alpha} T_1^{-j_\alpha + 1} T_2^{-i_\alpha + 1} \\ W &= \sum_{\alpha=1}^n T_{a_\alpha} \end{aligned} \tag{2.9}$$

It is then possible to compute the character as [2, 3]

$$\begin{aligned} \chi &= V^* \times V \times [T_1 + T_2 - T_1 T_2 - 1] + W^* \times V + V^* \times W \times T_1 T_2 \\ &= \sum_{\alpha, \beta}^n \sum_{s \in Y_j} \left( T_{a_{\alpha\beta}} T_1^{-h_\beta(s)} T_2^{v_\alpha(s)+1} + T_{a_{\beta\alpha}} T_1^{h_\beta(s)+1} T_2^{-v_\alpha(s)} \right) \end{aligned} \tag{2.10}$$

with  $a_{\alpha\beta} = a_\alpha - a_\beta$ .  $h_\beta(s)$  ( $v_\alpha(s)$ ) is the horizontal (vertical) distance from  $s$  till the right (top) end of the  $\alpha(\beta)$  diagram, i.e. the number of black (white) circles in Fig.1. See

Appendix C of [3] for a more detailed explanation of the computation and meaning of (2.10). The exponents in (2.10) are the eigenvalues of the operator  $\mathcal{L}_{x_0}$  which enters in our localization formula (2.6). Using these eigenvalues and (2.6), the partition function of  $\mathcal{N} = 2$  SYM for winding number  $k$  is [2, 3]

$$\mathcal{Z}_k = \sum_{x_0} \frac{1}{\det \hat{\mathcal{L}}_{x_0}} = \sum_{\{Y_\alpha; \sum_\alpha |Y_\alpha| = k\}} \prod_{\alpha, \beta=1}^n \prod_{s \in Y_\alpha} \frac{1}{E_{\alpha\beta}(s)(\epsilon - E_{\alpha\beta}(s))} \quad (2.11)$$

and

$$E_{\alpha\beta}(s) = a_{\alpha\beta} - \epsilon_1 h_\beta(s) + \epsilon_2 (v_\alpha(s) + 1) \quad (2.12)$$

$\mathcal{N} = 2^*$  SYM is obtained from pure  $\mathcal{N} = 4$  SYM giving mass to the adjoint  $\mathcal{N} = 2$  hypermultiplet. In this latter case the character is [3]

$$\chi_m = (1 - T_m^{-1})\chi \quad (2.13)$$

where  $\chi$  is the character of pure  $\mathcal{N} = 2$  SYM which was defined in (2.10).  $T_m = e^{im}$  parameterizes the mass deformation. In fact, a mass deformation can be introduced in a way similar to the equivariant deformation  $T_{\epsilon_{1,2}}$ . While  $T_{\epsilon_{1,2}}$  is embedded in the  $SU(2) \times SU(2)$  acting on Euclidean space time,  $T_m = e^{im}$  is a  $SO(2)$  subgroup of the  $SO(6)$   $\mathcal{R}$ -symmetry group which acts on the  $\mathbb{R}^6$  space transverse to the D3 branes we introduced before. In this case the partition function for winding number  $k$  is given by [3]

$$\mathcal{Z}_k = \sum_{\{Y_\lambda\}} \prod_{\lambda, \tilde{\lambda}=1}^N \prod_{s \in Y_\lambda} \frac{(E_{\alpha\beta}(s) - m)(E_{\alpha\beta}(s) - \epsilon + m)}{E_{\alpha\beta}(s)(E_{\alpha\beta}(s) - \epsilon)}. \quad (2.14)$$

The results for pure  $\mathcal{N} = 2$  SYM are easily obtained from (2.14) in the limit in which the mass of the hypermultiplet decouples, i.e.  $m \rightarrow \infty, m^4 q = \Lambda$ .

For simplicity we take  $\epsilon_2 = -\epsilon_1 = \hbar$ . It is convenient to introduce

the notation

$$\begin{aligned} f(x) &= \frac{(x - m)(x + m)}{x^2} \\ T_\alpha(x) &= \prod_{\beta \neq \alpha} \frac{(a_{\alpha\beta} + x + m)(a_{\alpha\beta} + x - m)}{(a_{\alpha\beta} + x)^2} \end{aligned} \quad (2.15)$$



The contributions coming from the first few tableaux can then be written as [3]

$$\begin{aligned}
Z_{\square} &= \sum_{\alpha} f(\hbar) T_{\alpha} \\
Z_{\square\square} &= \frac{1}{2} \sum_{\alpha \neq \beta} T_{\alpha} T_{\beta} f(a_{\alpha\beta} + \hbar) f(a_{\alpha\beta} - \hbar) \frac{f(\hbar)^2}{f(a_{\alpha\beta})^2} \\
Z_{\square\square\square} &= \sum_{\alpha} f(\hbar) f(2\hbar) T_{\alpha} T_{\alpha}(\hbar)
\end{aligned} \tag{2.16}$$

with  $T_{\alpha} = T_{\alpha}(0)$  and  $Z_{\square\square}$  given by  $Z_{\square\square}$  with  $\hbar \rightarrow -\hbar$ . The multi instanton partition function  $Z(q) = \sum_k Z_k q^k$  determines the prepotential  $\mathcal{F}(q)$  via the relation

$$\mathcal{F}(q) \equiv \lim_{\hbar \rightarrow 0} \mathcal{F}(q, \hbar) = \lim_{\hbar \rightarrow 0} \hbar^2 \ln Z(q) \tag{2.17}$$

The general function  $\mathcal{F}(q, \hbar)$  encodes the gravitational corrections to the  $\mathcal{N} = 2$  superpotential.

## 2.2 The construction of Kronheimer and Nakajima

In this section we review the construction of gauge instantons on ALE spaces [20]. ALE spaces can be obtained from the minimal resolution of orbifolds of the type  $\mathbb{R}^4/\Gamma$ . In terms of the D-brane construction we discussed in the introduction, the orbifold quotient is taken along the directions longitudinal to the D3-brane system. These directions form a  $\mathbb{R}^4$  space acted upon by the Lorentz group  $SO(4) \cong SU(2)_L \times SU(2)_R$ .  $\Gamma$  is a discrete Kleinian subgroup of  $SU(2)$ , i.e.  $\Gamma = \mathbb{Z}_p, D_N^*, O^*, T^*, I^*$ . The explicit computations in the next section will be carried out for the  $\mathbb{Z}_p$  case only. The recipe to get an ALE space is simple: take a pair of  $|\Gamma| \times |\Gamma|$  complex matrices  $\alpha, \beta$  satisfying the  $\Gamma$  invariance property

$$\gamma_v \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \gamma_v^{-1} = \gamma_Q \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \gamma \in \Gamma \tag{2.18}$$

where  $\gamma_v \in U(|\Gamma|)$  and  $\gamma_Q \in SU(2)$  are matrices realizing the element  $\gamma \in \Gamma$  and  $(\alpha, \beta)$  transforming in the adjoint of  $U(|\Gamma|)$  and in the fundamental of  $SU(2)$ . We then introduce a manifold  $\Xi$  with coordinates  $\Xi = (\alpha, \beta)$  of real dimension

$$\dim \Xi = 2 \sum_{i,j=1}^{|\Gamma|} A_{ij} m_i m_j = 4 \sum_{i=1}^{|\Gamma|} (m_i)^2 = 4|\Gamma| \tag{2.19}$$

where  $m_i$  are the dimensions of the irreducible representations,  $R_i$  in the decomposition  $\Gamma = \sum_{i=1}^{|\Gamma|} m_i R_i$ ,  $A_{ij} = 2\delta_{ij} - \tilde{C}_{ij}$  with  $\tilde{C}_{ij}$  the extended Cartan matrix connected to  $\Gamma$ . See [18] for an exhaustive description of these points. Starting from the datum  $\Xi$ , it is possible to build a two form analogous to (2.3), invariant under  $U(|\Gamma|)$  transformations. This invariance leads to two conserved quantities and to the constraints

$$\begin{aligned} [\alpha, \beta] &= \zeta_{\mathbb{C}} \\ [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] &= \zeta_{\mathbb{R}} \end{aligned} \quad (2.20)$$

where  $\zeta \in \mathbb{R}^3 \otimes \mathcal{Z}^*$ , with  $\mathcal{Z}^*$  the dual to the center of the Lie algebra of  $G = \otimes_{i=1}^{p-1} U(m_i)$ ,<sup>1</sup> with  $\sum_{i=1}^{p-1} \zeta_i = 0$ . Taking the quotient by  $G$  we finally get a manifold of dimension  $\dim X_\zeta = \dim \Xi - 4 \dim G = 4|\Gamma| - 4(|\Gamma| - 1) = 4$ .

On these spaces it is then possible to extend the ADHM construction of the previous subsection. With respect to what we already said, there are two further requests which need to be satisfied

- The matrix  $\Delta$  in (2.1) needs to be invariant under the action of  $\Gamma$ . The projection is acting on the Lorentz indices as  $\gamma_Q \in T_\epsilon$  and on Chan-Paton indices  $V, W$  as  $\gamma_v, \gamma_w$ .
- The instanton solution is classified by both the first and second Chern classes. To properly define them we introduce a tautological bundle  $\mathcal{T}$  with fiber the regular representation of  $\Gamma$  and base the ALE space itself.  $\mathcal{T}$  keeps in account the fact that parallel transporting a section of the bundle at infinity gives a holonomy due to the non trivial topology of the base space.

Under the action of  $\Gamma$  this tautological bundle admits a decomposition  $\mathcal{T} = \sum_q \mathcal{T}_q \otimes R_q$  with  $R_q$  ( $q = 0, 1, \dots, p-1$ ) the irreducible representations of  $\Gamma$ . The first Chern Class  $c_1(\mathcal{T}_q)$  of the  $\mathcal{T}_q$  bundles,  $q \neq 0$  ( $c_1(\mathcal{T}_0) = 0$ ), forms a basis of the second cohomology group. Under  $\Gamma$  we get the decompositions

$$V = \sum_q V_q \otimes R_q, \quad W = \sum_q W_q \otimes R_q \quad Q = Q_1 + Q_2$$

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<sup>1</sup> $G$  is obtained from  $U(|\Gamma|)$  keeping in account the previous decomposition of  $\Gamma$ . In its decomposition we have omitted  $U(m_0)$  whose action is trivial.

$$\gamma_{v,w} R_q = e^{\frac{2\pi i q}{p}} R_q \quad \gamma_Q Q_1 = e^{\frac{2\pi i}{p}} Q_1 \quad \gamma_Q Q_2 = e^{-\frac{2\pi i}{p}} Q_2 \quad (2.21)$$

The winding number  $k = \sum_q k_q$  and the rank of the bundle  $n = \sum_q n_q$  are given in terms of the integers  $k_q = \dim V_q, n_q = \dim W_q$ .<sup>2</sup>

The decompositions (2.21) determines that of the ADHM variables  $I, J, B_l$ . In particular the  $\Gamma$ -invariant components are

$$m_\Gamma = \{I_{I_q \alpha_q}, J_{\alpha_q I_q}, B_{1(I_{q+1} J_q)}, B_{2(I_{q-1} J_q)}\} \quad (2.22)$$

A similar result can be obtained for the fermions in the theory [17]. Notice that supersymmetry is preserved by this projection since  $\Gamma$  acts in the same way on the different components of a given supermultiplet.

The moduli space of multi instanton solutions and ADHM constraints can be described as before through (2.1), (2.5) written in terms of the invariant components (2.22). In particular the dimension of the moduli space is given by the total number of components in (2.22), minus the number of ADHM constraints (given by  $3k_q^2$  for each  $q$ ), minus the dimension of  $\prod_q U(k_q)$

$$\dim \mathcal{M} = 4 \sum_q (k_q n_q + \frac{1}{2} k_{q+1} k_q + \frac{1}{2} k_{q-1} k_q - k_q^2) \quad (2.23)$$

The decomposition properties can be used to relate the Chern character of the instanton bundle to the Chern characters of the individual bundles  $\mathcal{T}_q$ . The instanton bundle is specified then by giving the first and second Chern class

$$\begin{aligned} c_1 &= \sum_q (n_q - 2k_q + k_{q+1} + k_{q-1}) c_1(\mathcal{T}_q) \\ c_2 &= \sum_q (n_q - 2k_q + k_{q+1} + k_{q-1}) c_2(\mathcal{T}_q) + \frac{k}{|\Gamma|} \end{aligned} \quad (2.24)$$

The restriction to the interesting case of instanton solutions with vanishing first Chern class imposes, as we will see, strong constraints on the allowed instanton configurations  $\{k_q\}$  for a given partition  $\{n_q\}$ . In particular an instanton solution with vanishing first

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<sup>2</sup>The reader should not confuse this decomposition with the partitions leading to the Young tableaux we introduce in the previous subsection. The different notation emphasizes this fact.

Chern class is given by

$$n_q - 2k_q + k_{q+1} + k_{q-1} = 0 \quad \text{for } q > 0. \quad (2.25)$$

This is a highly non trivial constraint on the allowed values of  $(k_q, n_q)$ . Notice that only in this case the instanton number defined as  $k/|\Gamma| = k/p$  coincides with the second Chern class.

In [18] the reader will be able to find detailed discussions of various cases. Here we simply recall the results in the simplest context: the  $SU(2)$  gauge bundle on the Eguchi-Hanson blown down space  $\mathbb{R}^4/\mathbb{Z}_2$ . Solutions to (2.25) in this case are given by either  $\vec{n} = (2, 0), \vec{k} = (k, k)$  or  $\vec{n} = (0, 2), \vec{k} = (k - 1, k)$ . They lead to instanton solutions with integer and half-integer second Chern class respectively, as can be easily seen from (2.24). The dimension of the multi instanton moduli space can be read from (2.23) and turns out to be respectively equal to  $8k$  and  $8k - 4$ , in agreement with [18]. In the next section we derive again these results in a more general perspective and derive the corresponding  $\mathcal{N} = 2$  prepotential describing the low energy physics in the ALE.

### 3 Cohomology of moduli spaces

Localization is a powerful tool in the study of cohomology. Here we apply this techniques to various instanton moduli spaces. The basic idea is to associate a perfect Morse function, i.e. a function  $f(M)$  such that the part of the manifold  $M$  satisfying  $f \leq c$  for arbitrary positive  $c$  is compact, to a given momentum map defined by the action of a group  $G$  on  $M$ . Non-trivial  $p$ -cycles in  $M$  are then in one-to-one correspondence to the number of critical points  $\partial_i f(x_0) = 0$  with  $p$ -negative eigenvalues for the Hessian  $\partial_i \partial_j f(x_0)$  [23].

#### 3.1 $U(n)$ gauge theory on $\mathbb{R}^4$

Let us start by considering the Poincaré polynomial for the moduli space of  $U(n)$  gauge connections of winding number  $k$ ,  $\mathcal{M}_k^n$ :  $P_t(\mathcal{M}_k^n) = \sum_{n \geq 0} t^n \dim H^n(\mathcal{M}_k^n)$ <sup>3</sup>. First we ob-

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<sup>3</sup>The case of  $SU(n)$  is given by choosing  $\sum_{\alpha=1}^n a_\alpha = 0$ . This corresponds to factorize the center of mass of the brane system. For the computations we carry on in this paper this difference is immaterial.

serve that the momentum map obtained by acting on (2.3) with the Lie derivative with respect to the action of  $T_\epsilon^2$  and  $T_a$

$$M(B_l, I, J) = \sum_{\alpha=1}^n \left\{ \sum_{l=1}^2 \sum_{i,j=1}^{k_\alpha} \epsilon_l |(B_l)_{ij}|^2 + \sum_{i=1}^{k_\alpha} (a_\alpha |I_{i\alpha}|^2 + (\epsilon_1 + \epsilon_2 - a_\alpha) |J_{\alpha i}|^2) \right\} \quad (3.1)$$

is a perfect Morse function [19]. In fact choosing  $\epsilon_1 \gg a_n \gg \dots \gg a_1 \gg \epsilon_2 > 0$  one fulfills the condition that the part of  $\mathcal{M}$  satisfying  $\mathcal{M} \leq c$  for arbitrary positive  $c$  is compact. From the point of view of the brane system, the above choice means that the D(-1) instantons (whose position in the transverse space is given by (2.8)) are far enough ( $\epsilon_1 \gg a_\alpha$ ) to feel the whole  $U(n)$  stack of D3 branes. This choice also insures that the D(-1) instantons never sit on top of each other to avoid a singularity of the moduli space.

The Betti's numbers  $b_{2q} = \dim H^{2q}(\mathcal{M})$  (remember we are discussing a complex case) are given by the sum of all those Young tableaux  $Y_\alpha$  with  $q$  negative eigenvalues. Examining the character  $\chi$  of (2.10) we find the negative eigenvalues of the Hessian. They satisfy the conditions [19]

- $h_\beta(s) > 0$
- $h_\beta(s) = 0$  but  $\alpha < \beta$

The case  $h_\beta(s) = 0, \alpha = \beta, 1 + v_\alpha(s) < 0$ , never happens.

At each row there is a single box with  $h(s) = 0$  and therefore the number of boxes in  $Y_\alpha$  with  $h(s) = 0$  is the number of rows  $l_\alpha$ . The total number of negative eigenvalues in the Young tableaux pair  $(Y_\alpha, Y_\beta)$  is then given by

$$\sum_{\alpha, \beta} (k_\alpha - l_\alpha) + \sum_{\alpha < \beta} l_\alpha = nk + \sum_{\alpha=1}^n (\alpha - 1 - n) l_\alpha. \quad (3.2)$$

$k_\alpha$  is the number of boxes in the  $\alpha$ -th diagram. The generating function for the Poincaré polynomials is then

$$\begin{aligned} \sum_{k=0}^{\infty} q^k P_t(\mathcal{M}_k^n) &= \sum_{k=0}^{\infty} \sum_{\{Y_\alpha; |Y_\alpha|=k\}} q^k t^{2(nk + \sum_{\alpha} (\alpha-1-n) l_\alpha)} \\ &= \prod_{\alpha=1}^n \sum_{n_1, n_2, \dots} \prod_{m=1}^{\infty} \left( q^m t^{2(n(m-1) + \alpha - 1)} \right)^{n_m} \end{aligned}$$

$$= \prod_{\alpha=1}^n \prod_{m=1}^{\infty} \frac{1}{1 - q^m t^{2(n(m-1)+\alpha-1)}}. \quad (3.3)$$

In writing this we use the standard trick to rewrite  $k = \sum_{\alpha} \sum_{m=1}^{l_{\alpha}} m n_m^{\alpha}$ ,  $l_{\alpha} = \sum_{m=1}^{l_{\alpha}} 1$  and exchange the constrained sums with unconstrained ones over the  $n_1^{\alpha}, n_2^{\alpha}, \dots$ .  $n_i^{\alpha}$  is the number of times the integer  $i$  appears in the partition of  $k_{\alpha}$ .

A different choice of the relation between the parameters  $\epsilon_{1,2}$  and the v.e.v.'s  $a_{\alpha}$  leads to a different physical situation. Choosing  $a_n \gg \dots \gg a_1 \gg \epsilon_1 \gg \epsilon_2 > 0$  corresponds to D(-1) instantons living in well separated D3 branes and the Poincaré polynomial reduces to the one of the case  $U(1)^n$ .

### 3.2 Instantons on $\mathbb{R}^4/\mathbb{Z}_p$

Next we consider instanton on  $\mathbb{R}^4/\Gamma$  with  $\Gamma = \mathbb{Z}_p$ . The homologies of this space are the same of those for the manifold in which the conical singularity is simply resolved. The orbifold action is chosen to be

$$a_{\alpha} \rightarrow a_{\alpha} + \frac{2\pi q_{\alpha}}{p}, \quad \epsilon_1 \rightarrow \epsilon_1 + \frac{2\pi}{p}, \quad \epsilon_2 \rightarrow \epsilon_2 - \frac{2\pi}{p}. \quad (3.4)$$

where  $q_{\alpha}$ ,  $\alpha = 1, \dots, n$  can take integer values between 0 and  $p-1$ , specifying the representations under which the subspace  $W_{\alpha} \in W$  transform. In particular the integers  $n_q$  characterizing the unbroken gauge group  $\prod_q U(n_q)$  are given by the number of times that the  $q^{\text{th}}$ -representation appears in  $W$ , i.e.  $n_q = \sum_{\alpha} \delta_{q,q_{\alpha}}$ . Similarly the ADHM auxiliary group  $U(k)$  breaks into  $\prod_q U(k_q)$  with  $k_q$  the number of instantons transforming in the representation  $R_q$ , i.e.  $k_q = \dim V_q$ . According to (3.4) the instanton associated to the box  $(i_{\alpha}, j_{\alpha})$  in the tableaux  $Y_{\alpha}$  transforms in the representation  $R_{q_{\alpha}+i-j}$ . Take for example the following tableaux for a  $U(4)$  gauge theory on  $\mathbb{R}^4/\mathbb{Z}_3$

$$\begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 0 & 1 & 2 & \\ \hline 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 0 \\ \hline \end{array} \quad \begin{array}{c} 1 \\ \bullet \end{array} \quad \begin{array}{c} 2 \\ \bullet \end{array} \quad (3.5)$$

The number in the box in the bottom left position in the Young tableaux in (3.5) gives the representation,  $R_q$ , in which each D3 brane transforms. The bullet stands for an

empty Young tableaux which anyway transforms under  $\mathbb{Z}_3$ .<sup>4</sup> From (3.4) and (3.5) we then infer  $q_1 = 0, q_2 = 2, q_3 = 1, q_4 = 2$  i.e.  $n = (1, 1, 2)$ . The number of instantons in each representation follows then by counting the number of boxes in the Young tableaux labelled by the same integers  $q_\alpha$ . For the example of (3.5) we get  $k = (6, 6, 7)$ . The multi instanton moduli space splits into disconnected pieces  $\mathcal{M}_{(k_q, n_q)}$  specified by the set of integers  $(n_q, k_q)$ . As we will see these components are simply connected, i.e. for each component we get  $b_0 = 1$ , and support in general non-trivial cohomologies.

It is important to notice that  $\Gamma$  given by (3.4) belongs to the ADHM symmetry group  $G = U(1)^2 \otimes U(1)^n$  used for localization. This implies that a fixed point under  $G$  is automatically invariant under  $\Gamma$  and therefore critical points on the ALE instanton moduli space are given again in terms of  $n$ -sets of Young-tableaux with a total number of  $k$  boxes. Yet the contribution of each diagram to the determinant  $\det \mathcal{L}_{x_0}$  in (2.6) will be substantially different since only  $\Gamma$ -invariant boxes are now contributing. The same holds for the cohomology where the number of  $\Gamma$ -invariant negative eigenvalues of the Hessian at a given critical point will be in general smaller than the number of eigenvalues in the related flat space.

In addition the  $\Gamma$ -invariant analog of (2.10) is

$$\chi_\Gamma = \sum_q [V_q^* V_{q+1} + V_{q+1}^* V_q - V_q^* V_q \Lambda^2 Q - V_q^* V_q + W_q^* V_q + V_q^* W_q \Lambda^2 Q] \quad (3.6)$$

with  $\Lambda^2 Q = Q_1 Q_2$ . The character (3.6) is given by

$$\chi_\Gamma = \sum_{\alpha, \beta}^n \sum_{s \in Y_\alpha} \left( T_{a_{\alpha\beta}} T_1^{-h_\beta(s)} T_2^{v_\alpha(s)+1} + T_{a_{\beta\alpha}} T_1^{h_\beta(s)+1} T_2^{-v_\alpha(s)} \right) \delta_{h_\beta(s)+v_\alpha(s)+1, q_\alpha - q_\beta} \quad (3.7)$$

As in the case of  $\mathbb{R}^4$  we associate a non-trivial element of the  $2m$ -cohomology group of moduli space of instanton in  $\mathbb{R}^4/\mathbb{Z}_p$  to each Young tableaux with  $m$   $\Gamma$ -invariant negative eigenvalues in (3.7), i.e. the number of boxes with  $v_\alpha(s) + h_\beta(s) + 1 = q_\alpha - q_\beta \bmod p$  and  $h(s) > 0$ .

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<sup>4</sup>In the example of (3.5) this means that there are no D(-1) branes superposed to the third and fourth D3 brane.

As an illustration consider  $U(1)$  instantons on  $R^4/\mathbb{Z}_3$  with  $n = (1, 0, 0)$

[illegible]

Bullets and crosses stand for  $\Gamma$ -invariant boxes with  $h(s) > 0$  and  $h(s) = 0$  respectively. Counting the number of tableaux with a fixed number of bullets in (3.8) we find that the  $k = 6$   $U(1)$  instanton moduli space with  $n = (1, 0, 0)$  on  $R^4/\mathbb{Z}_3$  splits into three simply connected ( $b_0 = 1$ ) pieces which contribute to the Poincaré polynomial as

$$\begin{aligned} q^6 r^{2R_0+2R_1+2R_2} (1+3t^2+5t^4) : \quad & b_0 = 1 \quad b_2 = 3 \quad b_4 = 5 \\ q^6 r^{3R_0+R_1+2R_2} : \quad & b_0 = 1 \\ q^6 r^{3R_0+2R_1+R_2} : \quad & b_0 = 1 \end{aligned} \tag{3.9}$$

with  $q = e^{2\pi i\tau}$ . The factor  $r^{k_q R_q}$  describes the  $\Gamma$ -content.

## $U(1)$ instantons

Here we consider arbitrary instanton configurations for the  $U(1)$  case. For concreteness we take  $n = (1, 0, \dots, 0)$  and compute the Poincaré polynomial as a series  $\sum_{m, k_q} b_m(\mathcal{M}_{k_q}) q^{\sum_q k_q} t^m r^{k_q R_q}$ . Let us start by considering  $R^4/\mathbb{Z}_2$ . The Poincaré polynomial of multi instanton moduli space on  $R^4/\mathbb{Z}_2$  follows from that on  $R^4$  after decomposition in classes  $k_0 R_0 + k_1 R_1$

$$\begin{aligned}
Y &= \bullet + \square + \left( \begin{array}{|c|} \hline \bullet \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|} \hline \bullet & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \times \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|c|} \hline \bullet & \bullet & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \times \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} \right) + \dots \\
&= 1 + q^{rR_0} + (1+t^2)q^2 r^{R_0+R_1} + q^3(r^{2R_0+R_1}(1+t^2) + r^{R_0+2R_1}) + \\
&\quad + q^4 r^{2R_0+2R_1}(1+2t^2+2t^4) + \dots
\end{aligned} \tag{3.10}$$

There are two particularly interesting classes of diagrams entering in (3.10). The first one will be associated to the contribution of regular instantons and can be found by restricting



to those Young tableaux transforming in the regular representation of  $\mathbb{Z}_2$

$$\begin{aligned}
Y_{\text{reg}}^{\mathbb{Z}_2} &= \bullet + \left( \begin{array}{|c|c|} \hline \bullet & \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|c|} \hline \bullet & \bullet & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|} \hline \times & \times \\ \hline \end{array} + \begin{array}{|c|c|} \hline \times & \bullet \\ \hline \end{array} + \begin{array}{|c|} \hline \times \\ \hline \end{array} \right) + \dots \\
&= 1 + (1 + t^2)q^2 r^{R_0+R_1} + q^4 r^{2R_0+2R_1} (1 + 2t^2 + 2t^4) + \dots \quad (3.11)
\end{aligned}$$

This is the same Poincaré polynomial one would have obtained by studying the symmetric product  $S^k(R^4/\mathbb{Z}_2)$  [25]. The equivalence between the Poincaré polynomial of regular instantons and that of symmetric products holds for all the group  $\mathbb{Z}_p$  we have studied <sup>5</sup>. In mathematical terms this translates into the relation  $(\mathbb{C}^{[k]})^\Gamma = (\mathbb{C}/\Gamma)^{[k]}$  with  $M^{[k]}$  the Hilbert scheme of  $k$  points on  $M$ . More generally

$$\begin{aligned}
Y_{\text{reg}}^{\mathbb{Z}_p} &= \sum_k q_{\text{reg}}^k Y(S_k(R^4/\mathbb{Z}_p)) \\
&= \prod_{m=1}^{\infty} \frac{1}{(1 - q_{\text{reg}}^m t^{2m-2})(1 - q_{\text{reg}}^m t^{2m})^{p-1}} \quad (3.12)
\end{aligned}$$

with the parameter  $q_{\text{reg}} \equiv q^p r^{\text{Reg}}$  tracing the number of instanton in the regular representation  $R_{\text{reg}} \equiv \sum_q R_q$ . The second class is associated to fractional instantons and corresponds to tableaux with no  $\Gamma$ -invariant boxes (i.e with no bullets or crosses)

$$\begin{aligned}
Y_{\text{frac}}^{\mathbb{Z}_2} &= \bullet + \square + \begin{array}{|c|c|} \hline & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} + \dots \\
&= 1 + q r^{R_0} + q^3 r^{R_0+2R_1} + q^6 r^{4R_0+2R_1} + \dots \quad (3.13)
\end{aligned}$$

We call this class fractional to emphasize that this contributions carry no moduli (no bullets or crosses) i.e. they correspond to disjoint points (zero dimensional surfaces) in the moduli space. It is easy to find a closed form for this generating function. Let us consider separately the two cases in which the number of boxes in the first row are even or odd. In the former case the number of boxes in the first row is  $2k$  with  $k = 0, 1, \dots$  and the number of boxes transforming according to the  $R_0, R_1$  representations is  $k^2$  and  $k^2 + k$  respectively. In the latter case, if the number of boxes is  $2k - 1$  with  $k \in \mathbb{Z}_+$  the number of boxes transforming according to the  $R_0, R_1$  representations is  $k^2$  and  $k^2 - k$

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<sup>5</sup>We believe that this is always the case but we don't have a proof of this. In this paper we also report on the  $\mathbb{Z}_3$  case. We have also studied other cases with a Mathematica code.

respectively. Then

$$\begin{aligned}
Y_{\text{frac}}^{\mathbb{Z}_2} &= 1 + \sum_{k=1}^{\infty} \left( q^{k(2k-1)} r^{k^2 R_0 + (k^2 - k) R_1} + q^{k(2k+1)} r^{k^2 R_0 + (k^2 + k) R_1} \right) \\
&= \sum_{k=-\infty}^{\infty} (q r^{R_1})^k (q^2 r^{R_0 + R_1})^{k^2} = \theta_3(z_1 | q_{\text{reg}}^2)
\end{aligned} \tag{3.14}$$

with  $e^{2\pi i z_1} \equiv q r^{R_1}$  and  $q_{\text{reg}} \equiv q^p r^{R_{\text{reg}}}$  with  $R_{\text{reg}} = \sum_q R_q$ . Conventions for theta functions are explained in Appendix A. Each power of  $e^{2\pi i z_1}$  correspond to an unpaired fractional instanton in the  $R_1$ -representation of  $\mathbb{Z}_2$ . Remarkably the full instanton character  $Y$  can be written as the product of the regular  $Y_{\text{reg}}$  and fractional  $Y_{\text{frac}}$  contributions

$$Y = Y_{\text{reg}}^{\mathbb{Z}_2} Y_{\text{frac}}^{\mathbb{Z}_2} \tag{3.15}$$

as can be explicitly checked by taking the product of (3.12) and (3.14) and comparing against (3.10). The Poincaré polynomial can then be written as

$$P_t = \sum_{m,k,\mathcal{R}} b_m(\mathcal{M}_{k,\mathcal{R}}) q^k t^m r^{\mathcal{R}} = \prod_{n=1}^{\infty} \frac{(1 - q_{\text{reg}}^{2n})(1 + q r^{R_1} q_{\text{reg}}^{2n-1})(1 + (q r^{R_1})^{-1} q_{\text{reg}}^{2n-1})}{(1 - q_{\text{reg}}^n t^{2n-2})(1 - q_{\text{reg}}^n t^{2n})} \tag{3.16}$$

Similar relations hold for  $\mathbb{Z}_3$ . Now

$$\begin{aligned}
Y &= \bullet + \square + \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|c|} \hline \bullet & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|c|c|} \hline \square & \bullet & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} \right) + \dots \\
&= 1 + q r^{R_0} + q^2 (r^{R_0 + R_1} + r^{R_0 + R_2}) + q^3 r^{R_0 + R_1 + R_2} (1 + 2t^2) \\
&\quad + q^4 (r^{R_0 + 2R_1 + R_2} + r^{R_0 + R_1 + 2R_2} + (1 + 2t^2) r^{2R_0 + R_1 + R_2}) \dots
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
Y_{\text{reg}}^{\mathbb{Z}_3} &= \bullet + \left( \begin{array}{|c|c|c|} \hline \bullet & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} \right) + \dots \\
&= 1 + q^3 r^{R_0 + R_1 + R_2} (1 + 2t^2) + \dots \\
Y_{\text{frac}}^{\mathbb{Z}_3} &= \bullet + \square + \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + \left( \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \dots \\
&= 1 + q r^{R_0} + q^2 (r^{R_0 + R_1} + r^{R_0 + R_2}) + q^4 (r^{R_0 + R_1 + 2R_2} + r^{R_0 + 2R_1 + R_2}) + \dots
\end{aligned} \tag{3.18}$$

Again the relation (3.15) is verified. The extension of this algorithm to the case of  $U(n)$  is a straightforward matter.

### 3.3 The $\mathcal{N} = 4$ partition function on the Eguchi-Hanson manifold for gauge group $SU(2)$

In this last section of this chapter, we compute the Euler characteristic for the moduli space of gauge connections with half integer winding number on the Eguchi-Hanson manifold which is the simplest ALE variety. The Euler characteristic of the multi-instanton moduli space is the  $\mathcal{N} = 4$  partition function of the theory. Taking the limit of zero mass in the partition function (2.14) we in fact find that the  $\mathcal{N} = 4$  partition function is given by the sum over all critical points i.e. the Euler characteristic. With respect to the  $\mathbb{R}^4$  case [3] the novelty is that for the  $SU(2)$  Eguchi-Hanson case we have to consider only those critical points obeying to the conditions

$$\begin{aligned} q_1 = q_2 = 1 : \quad & k = (k_0, k_0 + 1) \quad n = (0, 2) \\ q_1 = q_2 = 0 : \quad & k = (k_0, k_0) \quad n = (2, 0) \end{aligned}$$

These conditions force us to count all the pairs of Young tableaux with an odd(even) number of boxes belonging to the classes  $k_0 R_{\text{reg}} + R_1(k_0 R_{\text{reg}})$  with the first box transforming according to the  $R_1(R_0)$  representation of  $\mathbb{Z}_p$ . The starting point is the  $U(2)$  Euler polynomial  $Y_{U(2)} = (Y_{\text{reg}}^{\mathbb{Z}_2} Y_{\text{frac}}^{\mathbb{Z}_2})^2|_{t=1}$  given by (3.12,3.14) i.e. before imposing the  $c_1 = 0$  condition. Singling out the classes  $k_0 R_{\text{reg}}$  and  $k_0 R_{\text{reg}} + R_1$  in this formula it is now easy. Unpaired instantons come only from

$$Y_{\text{frac}}^{\mathbb{Z}_2} = \sum_{k=-\infty}^{\infty} (qr^{R_1+q_1})^k q_{\text{reg}}^{k^2} \quad (3.19)$$

where we have modified the subscript of the term  $R_1$  to generalize (3.14) which was found in the case  $q_1 = 0$ <sup>6</sup>. Then

$$Y_{U(2)} = (Y_{\text{reg}}^{\mathbb{Z}_2})^2 \sum_{k_1, k_2=-\infty}^{\infty} (qr^{R_1+q_1})^{k_1} (qr^{R_1+q_2})^{k_2} (q_{\text{reg}})^{k_1^2+k_2^2}. \quad (3.20)$$

The factors  $(qr^{R_1+q_1})^{k_1}$  and  $(qr^{R_1+q_2})^{k_2}$  in (3.19) are the only potential sources of asymmetry between the  $R_0$  and  $R_1$  representations. The classes  $k_0 R_{\text{reg}}$  (with  $q_1 = q_2 = 0$ ) and

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<sup>6</sup>We remind the reader that the subscripts of the representations are always understood modulo  $p$ .

$k_0 R_{\text{reg}} + R_1$  (with  $q_1 = q_2 = 1$ ) are extracted from (3.20) by picking up the terms with  $k_1 = -k_2$  and  $k_1 = -k_2 - 1$  and specializing to  $r = 1$

$$\begin{aligned} Y_{\text{even}}^{SU(2)} &= q_{\text{reg}}^{-\frac{1}{6}} (Y_{\text{reg}}^{\mathbb{Z}_2})^2 \sum_{k=-\infty}^{\infty} q_{\text{reg}}^{2k^2} = \frac{\theta_3(0|q_{\text{reg}}^4)}{\eta(q_{\text{reg}})^4} \\ Y_{\text{odd}}^{SU(2)} &= q_{\text{reg}}^{-\frac{1}{6}} (Y_{\text{reg}}^{\mathbb{Z}_2})^2 \sum_{k=-\infty}^{\infty} q_{\text{reg}}^{\frac{4}{2}(k-\frac{1}{2})^2} = \frac{\theta_2(0|q_{\text{reg}}^4)}{\eta(q_{\text{reg}})^4} \end{aligned} \quad (3.21)$$

The factor  $q_{\text{reg}}^{-\frac{1}{6}}$  (Casimir energy) has been included as in [21] in order to reconstruct the modular functions. See the Appendix and (A.9) for our notation. The partition function of  $\mathcal{N} = 4$  is then

$$\mathcal{Z}_{\mathcal{N}=4} = q^{\frac{1}{6}} Y^{SU(2)} = \frac{1}{\eta^4(q_{\text{reg}})} [\theta_3(0|q_{\text{reg}}^4) + \theta_2(0|q_{\text{reg}}^4)] = \frac{\theta_3(0|\tau)}{\eta^4(\tau)} \quad (3.22)$$

Notice that the number of boxes in the diagrams is one half the Chern class (2.24). Therefore the correct expansion parameter for the  $\mathbb{R}^4/\mathbb{Z}_2$  case is  $q_{\text{reg}} = q^2$ . (3.22) is a modular form as expected from the  $SL(2, \mathbb{Z})$  duality of  $\mathcal{N} = 4$  [21].

## 4 The prepotential

Finally we derive the multi instanton partition functions and prepotentials describing the  $\mathcal{N} = 2$  low energy physics on the ALE manifold <sup>7</sup>. For simplicity we take  $\epsilon_2 = -\epsilon_1 = \hbar$ .

The ALE projection is defined by omitting the eigenvalues in (2.14) that are not invariant under the  $\mathbb{Z}_p$  projections

$$\mathbb{Z}_p : \hbar \rightarrow \hbar + \frac{2\pi}{p} \quad a_\alpha \rightarrow a_\alpha + \frac{2\pi q_\alpha}{p} \quad (4.1)$$

### $SU(2)$ gauge theory on $\mathbb{R}^4/\mathbb{Z}_2$

For concreteness we consider  $SU(2)$  gauge theory on  $\mathbb{R}^4/\mathbb{Z}_2$ . The condition of vanishing of the first Chern class (2.25),  $c_1 = 0$ , is only satisfied by the instanton configurations

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<sup>7</sup>In this chapter we will give the lowest terms in the expansion of the prepotential using an analytical method for pedagogical purposes. To check highest order terms, this way is not practical and we have written a Mathematica code.

[18]  $k = (k_0, k_0), n = (2, 0)(q_1 = q_2 = 0)$  and  $k = (k_0, k_0 + 1), n = (0, 2)(q_1 = q_2 = 1)$ . According to (2.24) they correspond to bundles with second Chern classes  $c_2 = k_0$  and  $c'_2 = \frac{1}{2}(2k_0 + 1)$  respectively.

The instanton partition function on the orbifold, follows from that in flat space after imposing invariance under (4.1). Thus only the eigenvalues (appearing as factors in (2.15)) invariant under (4.1) must be taken into account. In addition the ALE partition function is defined by

$$\mathcal{Z}(q) = \sum_k \mathcal{Z}_k q^{\frac{k}{2}}$$

since  $c_2 = \frac{1}{2}k$ .

Furthermore let us remark that (2.16) only depend on  $a_{\alpha\beta}$ . Since for instanton configurations with  $c_1 = 0$  one has  $q_\alpha = q_\beta$ , under (4.1) we get  $\delta a_{\alpha\beta} = \pi(q_\alpha - q_\beta) = 0$ . We now have to examine the behavior under  $\hbar \rightarrow \hbar + \pi$ . Only functions of the type  $f(n\hbar), T_\alpha(n\hbar)$  with  $n$  even survive this projection. Collecting these type of terms in (2.16) and setting  $a_1 = -a_2 = a$  one finds

$$Z_{\square} = 2 \left( 1 - \frac{m^2}{4a^2} \right) \quad (4.2)$$

$$Z_{\square\square} = Z_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}} = 2 \left( 1 - \frac{m^2}{(2\hbar)^2} \right) \left( 1 - \frac{m^2}{4a^2} \right) \quad (4.3)$$

The configuration  $Z_{\square\square}$  does not contribute, since it has  $c_1 \neq 0$ . Computing also terms which come from Young tableaux with three and four boxes we finally get

$$\begin{aligned} Z(q) = & 1 + \frac{(4a^2 - m^2)}{2a^2} q^{\frac{1}{2}} + \frac{(4a^2 - m^2)(4\hbar^2 - m^2)}{4a^2\hbar^2} q \\ & + \frac{m^2 a^2 (4a^2 - m^2) + \hbar^2 (128a^6 - 64m^2 a^4 + 8m^4 a^2 - 3m^6)}{16a^6 \hbar^2} q^{\frac{3}{2}} \\ & + \left[ \frac{4m^4 a^4 (-4a^2 + m^2) + \hbar^2 a^2 (-704a^6 m^2 + 352a^4 m^4 - 120a^2 m^6 + 5m^8)}{128a^8 \hbar^4} \right. \\ & \left. + \frac{\hbar^4 (2048a^8 - 1152m^2 a^6 + 640m^4 a^4 - 152a^2 m^6 + 9m^8)}{128a^8 \hbar^4} \right] q^2 + \dots \end{aligned} \quad (4.4)$$

From (4.4) we can extract the partition function for the pure  $\mathcal{N} = 2$  case by taking the limit  $m \rightarrow \infty, m^4 q = \Lambda$

$$Z(\Lambda) = 1 - \frac{\Lambda^{\frac{1}{2}}}{2a^2} + \frac{\Lambda}{4a^2\hbar^2} - \Lambda^{\frac{3}{2}} \frac{(a^2 + 3\hbar^2)}{8\hbar^2 a^6} + \Lambda^2 \frac{(4a^4 + 5a^2\hbar^2 + 9\hbar^4)}{128a^8\hbar^4} + \dots \quad (4.5)$$

The presence of half-integer powers of  $q$  makes  $Z(q)$  double-valued in the complex plane, i.e.  $Z(q)$  is not invariant under  $q^{\frac{1}{2}} \rightarrow -q^{\frac{1}{2}}$  or  $\tau \rightarrow \tau + 1$ . Out of  $Z(q)$  we can build two objects with definite transformation properties under  $\tau \rightarrow \tau + 1$ :

$$\begin{aligned}\mathcal{F}_{\text{even}}(q, m, \hbar) &= \frac{1}{2}\hbar^2 \left[ \ln Z(q^{\frac{1}{2}}) + \ln Z(-q^{\frac{1}{2}}) \right] \\ \mathcal{F}_{\text{odd}}(q, m, \hbar) &= -\frac{1}{2} \left[ \ln Z(q^{\frac{1}{2}}) - \ln Z(-q^{\frac{1}{2}}) \right]\end{aligned}\quad (4.6)$$

The two terms correspond to contributions with integer and half-integer Chern class respectively. Notice the different normalization of the two pieces. The  $\frac{1}{\hbar^2}$  volume factor in front of the odd term has been projected out by the  $\mathbb{Z}_2$  orbifold group action. Notwithstanding the odd term is now regular in the limit  $\hbar \rightarrow 0$ . The quadratic divergence is in fact connected to the breaking of translational invariance<sup>8</sup> in the lagrangian of the theory due to the rotations  $T_{\epsilon_{1,2}}$ . The lagrangian invariant under the infinitesimal action of  $T_{\epsilon_{1,2}}$  explicitly contains the modulus which gives the position of the center of the instanton. The integration over this modulus is responsible for the above mentioned quadratic divergence. The  $\mathbb{Z}_2$  invariant instanton is clearly centered in zero and there is no modulus for its center. For  $\mathcal{N} = 2^*$  one finds

$$\begin{aligned}\mathcal{F}_{\text{even}}(q) &= \left( \frac{m^4 - 4m^2a^2}{4a^2} \right) q + \frac{m^2}{128a^6} (5m^6 - 48m^4a^2 + 96m^2a^4 - 192a^6) q^2 \\ &\quad + \hbar^2 \left[ \left( \frac{16a^4 - m^4}{8a^4} \right) q + \left( \frac{-17m^8 + 72m^6a^2 - 128m^2a^6 + 512a^8}{128a^8} \right) q^2 \right] + \dots \\ \mathcal{F}_{\text{odd}}(q) &= \left( \frac{m^2 - 4a^2}{2a^2} \right) q^{\frac{1}{2}} - \left( \frac{32a^6 - 24m^2a^4 + 24m^4a^2 - 5m^6}{12a^6} \right) q^{\frac{3}{2}} \\ &\quad + \left( \frac{48m^2a^4 - 32m^4a^2 + 5m^6}{8a^8} \right) q^{\frac{5}{2}} + \dots\end{aligned}\quad (4.7)$$

Once again the pure  $\mathcal{N} = 2$  limit is recovered in the limit  $m \rightarrow \infty, m^4q = \Lambda$

$$\begin{aligned}\mathcal{F}_{\text{even}}(\Lambda) &= \left( \frac{1}{4} \frac{\Lambda}{a^2} + \frac{5}{128} \frac{\Lambda^2}{a^6} + \frac{3}{128} \frac{\Lambda^3}{a^{10}} \right) - \hbar^2 \left( \frac{1}{8} \frac{\Lambda}{a^4} + \frac{17}{128} \frac{\Lambda^2}{a^8} + \frac{13}{64} \frac{\Lambda^3}{a^{12}} \right) + \dots \\ \mathcal{F}_{\text{odd}}(\Lambda) &= \left( \frac{1}{2} \frac{\Lambda^{\frac{1}{2}}}{a^2} + \frac{5}{12} \frac{\Lambda^{\frac{3}{2}}}{a^6} + \frac{207}{320} \frac{\Lambda^{\frac{5}{2}}}{a^{10}} \right) + \hbar^2 \left( \frac{5}{8} \frac{\Lambda^{\frac{3}{2}}}{a^8} + \frac{269}{64} \frac{\Lambda^{\frac{5}{2}}}{a^{12}} \right) + \dots\end{aligned}\quad (4.8)$$

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<sup>8</sup>This translational symmetry of the lagrangian in the moduli space follows from the translational invariance of the instanton solution in Euclidean space time. In [1] it was suggested to deform the original lagrangian under  $T_{\epsilon_{1,2}}$  to allow for a localization with isolated critical points.

The  $k = 1/2, \hbar = 0$  term matches (4.14) in [26]<sup>9</sup>. For completeness we have also given the first gravitational correction to the prepotential.

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## A Theta functions

The conventions for theta functions in the text are:

$$\begin{aligned}\theta_{[b]}^{[a]}(z|q) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(z-b)(n-a)} \\ \eta(q) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\end{aligned}\tag{A.9}$$

where  $\theta_1 = \theta_{[\frac{1}{2}]}^{[\frac{1}{2}]}$ ,  $\theta_2 = \theta_{[0]}^{[\frac{1}{2}]}$ ,  $\theta_3 = \theta_{[0]}^{[0]}$ ,  $\theta_4 = \theta_{[\frac{1}{2}]}^{[0]}$ .

## B Cohomology of the ALE space

As an illustration here we compute the homologies of the ALE space itself. This is a well known result (see [24] for a review). Here we will derive it using the methods of subsection 2.1: the ALE space can be described in terms of the non commutative ADHM

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<sup>9</sup>In that reference the correlator  $Tr\phi^2$  is computed. For the lowest winding number this is the same as the prepotential.

formalism for the  $U(1)$  case. In fact we have seen in subsection 2.2 that the matrices which describe the ALE space are subject to the constraint (2.20) and to the  $\Gamma$  invariant condition (2.18). If in the ADHM construction we described in subsection 2.1 we take the  $U(1)$  case (i.e. the space  $W$  becomes a  $k$  dimensional vector), we can set  $J = 0$  and the constraints  $f_{\mathbb{C}} = 0, f_{\mathbb{R}} = \zeta$  coincide with (2.20)<sup>10</sup> Moreover choosing  $k = p$  in the regular representation the projection on ADHM moduli space matches that in (2.18). We can then compute the homologies of an ALE space using the character  $\chi$  in (3.7). The  $\Gamma$  invariance requires in this case  $h(s) + v(s) + 1 = p$ . How many boxes in the set of diagrams with  $|Y| = |\Gamma|$  obey this condition? The answer is given in Fig.B.

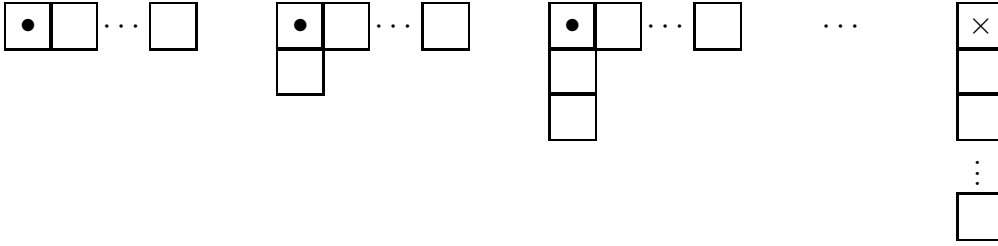


Figure 1: Subset of diagrams with  $p$  boxes satisfying  $h(s) + v(s) + 1 = p$ .

There we draw the only diagrams in which at least one box satisfies the  $\Gamma$  invariance condition. As before "•" refers to  $\Gamma$ -invariant boxes with  $h(s) > 0$  while "×" stands for  $h(s) = 0$ . Counting the number of bullets one finds the homology

- $b_0 = 1$  since the box denoted by a  $\times$  in the last diagram from the left in Fig.B satisfies the  $\Gamma$  invariance condition but has  $h(s) = 0$
- $b_2 = p - 1$  since there are  $p - 1$  diagrams with one negative eigenvalue, given by the boxes with a •.

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<sup>10</sup>The deformation appearing in (2.20) is subject to the condition  $\sum_{i=1}^{p-1} \zeta_i = 0$  while the  $\zeta$  in  $f_{\mathbb{R}} = \zeta$  is unconstrained. The two conditions agree if we take into account the term  $II^\dagger$  appearing in  $f_{\mathbb{R}}$ .



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